ART. XLI.—*Some Theorems relating to Sub-Polar Triangles.*

By EVELYN G. HOGG, M.A., Christ's College, Christchurch.

[*Read before the Philosophical Institute of Canterbury, 1st December, 1909.*]

§1. LET the straight lines joining the vertices of the triangle ABC to the points O_1, O_2 meet the opposite sides BC, CA, AB, in D_1D_2, E_1E_2, F_1F_2, respectively; then the triangles $D_1E_1F_1$, $D_2E_2F_2$ are termed sub-polar triangles, the points O_1, O_2 being their respective poles.*

The vertices of any two sub-polar triangles lie on a conic, for, taking the co-ordinates of the points O_1, O_2 to be $(\alpha_1 \beta_1 \gamma_1)$, $(\alpha_2 \beta_2 \gamma_2)$ respectively, it may be at once verified that the conic

$$\frac{\alpha^2}{\alpha_1 \alpha_2} + \frac{\beta^2}{\beta_1 \beta_2} + \frac{\gamma^2}{\gamma_1 \gamma_2} - \beta\gamma\left(\frac{1}{\beta_1\gamma_2} + \frac{1}{\beta_2\gamma_1}\right) - \gamma\alpha\left(\frac{1}{\gamma_1\alpha_2} + \frac{1}{\gamma_2\alpha_1}\right)$$
$$- \alpha\beta\left(\frac{1}{\alpha_1\beta_2} + \frac{1}{\alpha_2\beta_1}\right) = o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

passes through the vertices of $D_1E_1F_1$ and $D_2E_2F_2$.

If the points O_1, O_2 be regarded as being determined by the intersection of the line $L \equiv l\alpha + m\beta + n\gamma = o$ with the conic $S \equiv \alpha_o \beta\gamma + \beta_o \gamma\alpha + \gamma_o \alpha\beta = o$, then the conic (i) takes the form

$$\frac{l\alpha^2}{\alpha_o} + \frac{m\beta^2}{\beta_o} + \frac{n\gamma^2}{\gamma_o} + \frac{\beta\gamma}{\beta_o\gamma_o}(m\beta_o + n\gamma_o - l\alpha_o) + \frac{\gamma\alpha}{\gamma_o\alpha_o}(n\gamma_o + l\alpha_o - m\beta_o)$$
$$+ \frac{\alpha\beta}{\alpha_o\beta_o}(l\alpha_o + m\beta_o - n\gamma_o) = o \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

Let the line $L = o$ pass through the fixed point $O'(\alpha'\beta'\gamma')$; then, eliminating l between equation (ii) and the relation $l\alpha' + m\beta' + n\gamma' = o$, the conic (i) takes the form $mS_1 + nS_2 = o$, when

$$S_1 \equiv \beta'\frac{\alpha^2}{\alpha_o} - \alpha'\frac{\beta^2}{\beta_o} - (\alpha_o\beta' + \alpha'\beta_o)\frac{\gamma}{\gamma_o}\left(\frac{\alpha}{\alpha_o} - \frac{\beta}{\beta_o}\right) - \frac{\alpha\beta}{\alpha_o\beta_o}(\alpha_o\beta' - \alpha'\beta_o) = o,$$

$$S_2 \equiv \gamma'\frac{\alpha^2}{\alpha_o} - \alpha'\frac{\gamma^2}{\gamma_o} + (\gamma_o\alpha' + \gamma'\alpha_o)\frac{\beta}{\beta_o}\left(\frac{\gamma}{\gamma_o} - \frac{\alpha}{\alpha_o}\right) - \frac{\gamma\alpha}{\gamma_o\alpha_o}(\gamma'\alpha_o - \gamma_o\alpha') = o.$$

Hence we derive the theorem,—

If the poles of two sub-polar triangles be determined by the intersection of a variable line passing through a fixed point with a fixed conic circumscribing the triangle of reference, the conic circumscribing the two sub-polar triangles passes through four fixed points.

It will be seen on inspection that one of the points of intersection of the two conics S′, S″ is the point $(\alpha_o\beta_o\gamma_o)$—the pole of the conic S_o.

A particular case arises if we suppose the variable line L to pass through the point $(\alpha_o\beta_o\gamma_o)$. Conic (ii) reduces to

$$\frac{l\alpha^2}{\alpha_o} + \frac{m\beta^2}{\beta_o} + \frac{n\gamma^2}{\gamma_o} - 2l\alpha_o\frac{\beta\gamma}{\beta_o\gamma_o} - 2m\beta_o\frac{\gamma\alpha}{\gamma_o\alpha_o} - 2n\gamma_o\frac{\alpha\beta}{\alpha_o\beta_o} = o \quad . \quad . \quad \text{(iii)}$$

subject to the relation $l\alpha_o + m\beta_o + n\gamma_o = o$. By eliminating l we obtain

$$m\beta_o\left(\frac{\alpha}{\alpha_o} - \frac{\beta}{\beta_o}\right)\left(\frac{\alpha}{\alpha_o} + \frac{\beta}{\beta_o} + \frac{2\gamma}{\gamma_o}\right) - n\gamma_o\left(\frac{\gamma}{\gamma_o} - \frac{\alpha}{\alpha_o}\right)\left(\frac{\alpha}{\alpha^o} + \frac{2\beta}{\beta_o} + \frac{\gamma}{\gamma_o}\right) = o,$$

which shows that all conics of this family pass through the four fixed points $(\alpha_o\beta_o\gamma_o)$, $(-3\alpha_o\beta_o\gamma_o)$, $(\alpha_o - 3\beta_o\gamma_o)$, $(\alpha_o\beta_o - 3\gamma_o)$.

If the poles of two sub-polar triangles be the circular points at infinity, conic (i) takes the form

$$\alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma\cos A + 2\gamma\alpha\cos B + 2\alpha\beta\cos C = o.$$

If the two points O_1, O_2 be the extremities of a diameter of the circle ABC, then conic (i) takes the form

$$\lambda\frac{\alpha^2}{\sin A\cos A} + \mu\frac{\beta^2}{\sin B\cos B} + \nu\frac{\gamma^2}{\sin C\cos C}$$
$$+\frac{\beta\gamma}{\sin B\sin C}(\mu\tan B + \nu\tan C - \lambda\tan A)$$
$$+\frac{\gamma\alpha}{\sin C\sin A}(\nu\tan C + \lambda\tan A - \mu\tan B)$$
$$+\frac{\alpha\beta}{\sin A\sin B}(\lambda\tan A + \mu\tan B - \nu\tan C) = o,$$

where $\lambda + \mu + \nu = o$.

Since the equation of any diameter of the circle ABC is

$$\frac{\lambda\alpha}{\cos A} + \frac{\mu\beta}{\cos B} + \frac{\nu\gamma}{\cos C} = o,$$

if the diameter pass through the symmedian point

$$\lambda\tan A + \mu\tan B + \nu\tan C = o,$$

and the above equation reduces to

$$\frac{\sin(B-C)}{\sin A}\alpha^2 + \frac{\sin(C-A)}{\sin B}\beta^2 + \frac{\sin(A-B)}{\sin C}\gamma^2 - \frac{2\sin A\sin(B-C)}{\sin B\sin C}\beta\gamma$$
$$-\frac{2\sin B\sin(C-A)}{\sin C\sin A}\gamma\alpha - \frac{2\sin C\sin(A-B)}{\sin A\sin B}\alpha\beta = o,$$

a conic which passes through the symmedian point of the triangle ABC.

If O_1, O_2 be the extremities of a diameter of the Steiner ellipse

$$\frac{1}{a\alpha} + \frac{1}{b\beta} + \frac{1}{c\gamma} = o,$$

then the conic (i) reduces to

$$\lambda(a^2\alpha^2 - 2bc\beta\gamma) + \mu(b^2\beta^2 - 2ca\gamma\alpha) + \nu(c^2\gamma^2 - 2ab\alpha\beta) = o,$$

where $\lambda + \mu + \nu = o$.

§ 2. The condition that conic (i) should be a rectangular hyperbola is

$$\frac{1}{\alpha_1\alpha_2} + \frac{1}{\beta_1\beta_2} + \frac{1}{\gamma_1\gamma_2} + \cos A\left(\frac{1}{\beta_1\gamma_2} + \frac{1}{\beta_2\gamma_1}\right) + \cos B\left(\frac{1}{\gamma_1\alpha_2} + \frac{1}{\gamma_2\alpha_1}\right)$$
$$+\cos C\left(\frac{1}{\alpha_1\beta_2} + \frac{1}{\alpha_2\beta_1}\right) = o.$$

Hence, if $O_1(\alpha_1\beta_1\gamma_1)$ be fixed, the locus of O_2 will be the circum-conic

$$\frac{1}{\alpha}\left(\frac{1}{\alpha_1} + \frac{\cos C}{\beta_1} + \frac{\cos B}{\gamma_1}\right) + \frac{1}{\beta}\left(\frac{\cos C}{\alpha_1} + \frac{1}{\beta_1} + \frac{\cos A}{\gamma_1}\right)$$
$$+\frac{1}{\gamma}\left(\frac{\cos B}{\alpha_1} + \frac{\cos A}{\beta_1} + \frac{1}{\gamma_1}\right) = o \quad . \quad . \quad . \quad . \quad . \quad (iv).$$

If O_1 be the centroid of the triangle ABC, then the conic (iv) reduces to the circle ABC.

Hence the theorem,—

A rectangular hyperbola can be drawn through the middle points of the sides of a triangle and the points in which the sides of the triangle are cut by lines joining any point on the circum-circle of the triangle to the vertices of the triangle.

If O_1 be the orthocentre of the triangle ABC, then conic (iv) reduces to

$$\frac{\cos(B-C)}{\alpha} + \frac{\cos(C-A)}{\beta} + \frac{\cos(A-B)}{\gamma} = o.$$

§ 3. If the circle described about the triangle $D_1E_1F_1$ meet the sides BC, CA, AB of the triangle ABC again in the respective points $D^1E^1F^1$, then the lines AD^1, BE^1, CF^1 are concurrent. Let the co-ordinates of the point of concurrence O^1 be $(\alpha^1\beta^1\gamma^1)$. The vertices of the triangles $D_1E_1F_1$, $D^1E^1F^1$ lie on the circle

$$\frac{\alpha^2}{\alpha_1\alpha^1} + \frac{\beta^2}{\beta_1\beta^1} + \frac{\gamma^2}{\gamma_1\gamma^1} - \beta\gamma\left(\frac{1}{\beta_1\gamma^1} + \frac{1}{\beta^1\gamma_1}\right) - \gamma\alpha\left(\frac{1}{\gamma_1\alpha^1} + \frac{1}{\gamma^1\alpha_1}\right)$$
$$-\alpha\beta\left(\frac{1}{\alpha_1\beta^1} + \frac{1}{\alpha^1\beta_1}\right) = o \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v}).$$

From the conditions for a circle, we at once obtain

$$\frac{1}{\alpha^1} : \frac{1}{\beta^1} : \frac{1}{\gamma^1} = a\,(\gamma_1^2\alpha_1^2 + \alpha_1^2\beta_1^2 - \beta_1^2\gamma_1^2) - 2\alpha_1\beta_1\gamma_1\cos A\,(a\alpha_1 + b\beta_1 + c\gamma_1)$$
$$: b\,(\alpha_1^2\beta_1^2 + \beta_1^2\gamma_1^2 - \gamma_1^2\alpha_1^2) - 2\alpha_1\beta_1\gamma_1\cos B\,(a\alpha_1 + b\beta_1 + c\gamma_1)$$
$$: c\,(\beta_1^2\gamma_1^2 + \gamma_1^2\alpha_1^2 - \alpha_1^2\beta_1^2) - 2\alpha_1\beta_1\gamma_1\cos C\,(a\alpha_1 + b\beta_1 + c\gamma_1).$$

It may be at once verified that if $\alpha_1 : \beta_1 : \gamma_1 = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, then $\alpha^1 : \beta^1 : \gamma^1 = \sec A : \sec B : \sec C$, and the circle in this case is the nine-point circle of the triangle ABC.

If $a\alpha_1 + b\beta_1 + c\gamma_1 = o$, then

$$\kappa\left(\frac{1}{b\beta^1} + \frac{1}{c\gamma^1}\right) = \alpha_1^2,$$
$$\kappa\left(\frac{1}{c\gamma^1} + \frac{1}{a\alpha^1}\right) = \beta_1^2,$$
$$\kappa\left(\frac{1}{a\alpha^1} + \frac{1}{b\beta^1}\right) = \gamma_1^2,$$

hence the locus of the point $\alpha^1\beta^1\gamma^1$ is the quartic curve

$$a\sqrt{\left(\frac{1}{b\beta} + \frac{1}{c\gamma}\right)} + b\sqrt{\left(\frac{1}{c\gamma} + \frac{1}{a\alpha}\right)} + c\sqrt{\left(\frac{1}{a\alpha} + \frac{1}{b\beta}\right)} = o \quad . \quad . \quad . \quad (\text{vi}).$$

This result may be stated as follows:—

If lines drawn through the vertices of the triangle ABC parallel to a given line L meet the sides of that triangle in $D_1E_1F_1$, and the circle through $D_1E_1F_1$ intersect the sides again in $D^1E^1F^1$, then the lines AD^1, BE^1, CF^1 are concurrent in the point O^1, and as L turns about a fixed point in the plane ABC, the locus of O^1 will be the above curve (vi).

This quartic curve is the isogonal transformation of

$$\sqrt{a(b\gamma + c\beta)} + \sqrt{b(c\alpha + a\gamma)} + \sqrt{c(a\beta + b\alpha)} = o,$$

a conic inscribed in the triangle formed by drawing tangents to the circle ABC at the vertices of the triangle of reference.

The lines joining the points of contact of the conic with the sides of the circumscribing triangle to the opposite vertices of that triangle are concurrent in the point

$$\left[\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2},\ \frac{1}{c^2}+\frac{1}{a^2}-\frac{1}{b^2},\ \frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right].$$

If the point $O^1(\alpha^1\beta^1\gamma^1)$ lie on the circle ABC, then the locus of $O_1(\alpha_1\beta_1\gamma_1)$ is a quartic curve whose isogonal transformation is the conic $\alpha^2 \cot A + \beta^2 \cot B + \gamma^2 \cot C - 2(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) = o.$

If the point O^1 lie on the Steiner ellipse

$$\frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=o,$$

then the locus of O_1 is the quartic curve whose isogonal transformation is the conic

$$abc(\alpha^2+\beta^2+\gamma^2)-(a^2+b^2+c^2)(a\beta\gamma+b\gamma\alpha+c\alpha\beta)=o.$$

§4. The remainder of the paper is concerned with the cases arising when the poles of the two sub-polar triangles are isogonally conjugate with respect to the triangle ABC. Let a point $O(\alpha_o\beta_o\gamma_o)$ and its isogonal conjugate $O'\left(\frac{1}{\alpha_o}\frac{1}{\beta_o}\frac{1}{\gamma_o}\right)$ be taken, and let their sub-polar triangles be DEF, D′E′F′. Also let

$$p\equiv\frac{\beta_o}{\gamma_o}-\frac{\gamma_o}{\beta_o},\quad q\equiv\frac{\gamma_o}{\alpha_o}-\frac{\alpha_o}{\gamma_o},\quad r\equiv\frac{\alpha_o}{\beta_o}-\frac{\beta_o}{\alpha_o}$$

$$p'\equiv\frac{\beta_o}{\gamma_o}+\frac{\gamma_o}{\beta_o},\quad q'\equiv\frac{\gamma_o}{\alpha_o}+\frac{\alpha_o}{\gamma_o},\quad r'\equiv\frac{\alpha_o}{\beta_o}+\frac{\beta_o}{\alpha_o}.$$

The equation of the line Oo' is

$$L\equiv\alpha\alpha_o(\beta_o^2-\gamma_o^2)+\beta\beta_o(\gamma_o^2-\alpha_o^2)+\gamma\gamma_o(\alpha_o^2-\beta_o^2)=0,$$

which, after dividing out by $\alpha_o\beta_o\gamma_o$, reduces to

$$L\equiv p\alpha+q\beta+r\gamma=o.$$

The sub-polar triangles whose poles are isogonal conjugates are self-conjugate with respect to the conic which is the isogonal transformation of the line joining their poles.

The co-ordinates of the points D, D′ are respectively $(O\beta_o\gamma_o)$, $\left(O\frac{1}{\beta_o}\frac{1}{\gamma_o}\right)$, and the equations of EF and E′F′ are

$$-\frac{\alpha}{\alpha_o}+\frac{\beta}{\beta_o}+\frac{\gamma}{\gamma_o}=o$$

$$-\alpha\alpha_o+\beta\beta_o+\gamma\gamma_o=o.$$

The equations of the polars of D, D′ with respect to the conic

$$S_o\equiv p\beta\gamma+q\gamma\alpha+r\alpha\beta=o$$

are

$$p(\beta\gamma_o+\gamma\beta_o)+\alpha(q\gamma_o+r\beta_o)=o,$$

$$p(\beta\beta_o+\gamma\gamma_o)+\alpha(q\beta_o+r\gamma_o)=o,$$

which at once reduce to the equations found for EF, E′F′, and so prove the theorem.

The vertices of the triangles DEF, D′E′F′ lie on the conic

$$S \equiv \alpha^2 + \beta^2 + \gamma^2 - p'\beta\gamma - q'\gamma\alpha - r'\alpha\beta = o,$$

and the sides of the two triangles touch the conic

$$\Sigma \equiv \frac{\alpha^2}{pp'} + \frac{\beta^2}{qq'} + \frac{\gamma^2}{rr'} = o.$$

The polars of the points O, O′ with respect to the triangle ABC are

$$\frac{\alpha}{\alpha_o} + \frac{\beta}{\beta_o} + \frac{\gamma}{\gamma_o} = o,$$

$$\alpha\alpha_o + \beta\beta_o + \gamma\gamma_o = o,$$

and these lines meet in the point Ω whose co-ordinates are $(p\ q\ r)$.

Let the sides EF, FD, DF, E′F′, F′D′, D′E′ touch Σ in the points P, Q, R, P′, Q′, R′ respectively: the equations of PP′, QQ′, RR′ are

$$-\frac{\alpha}{p'} + \frac{\beta}{q'} + \frac{\gamma}{r'} = o,$$

$$\frac{\alpha}{p'} - \frac{\beta}{q'} + \frac{\gamma}{r'} = o,$$

$$\frac{\alpha}{p'} + \frac{\beta}{q'} - \frac{\gamma}{r'} = o.$$

Hence the triangle formed by PP′, QQ′, RR′ is the sub-polar triangle of the point Ω′ $(p'q'r')$.

The conic S is the harmonic conic of

$$\Sigma_1 \equiv \sqrt{\frac{\alpha}{\alpha_o}} + \sqrt{\frac{\beta}{\beta_o}} + \sqrt{\frac{\gamma}{\gamma_o}} = o$$

$$\Sigma_2 \equiv \sqrt{\alpha_o\alpha} + \sqrt{\beta_o\beta} + \sqrt{\gamma_o\gamma} = o.$$

The co-ordinates of intersection of the two conics Σ_1, Σ_2 are*

$$(p'-2,\ q'-2,\ r'-2) \quad (p'-2,\ q'+2,\ r'+2)$$

$$(p'+2,\ q'-2,\ r'+2) \quad (p'+2,\ q'+2,\ r'-2).$$

The lines joining these points are of the form

$$\alpha(q'-\gamma') - (p'-2)(\beta-\gamma) = o,$$

$$\alpha(q'+r') - (p'+2)(\beta+\gamma) = o;$$

and it is easily shown that the co-ordinates of the intersections of the joining lines not lying on Σ_1 and Σ_2 are $(-p\ q\ r)$, $(p\ -q\ r)$ $(p\ q\ -r)$. These are the points in which the corresponding sides of the triangles DEF, D′E′F′ meet. Hence the theorem,—

The intersections of corresponding sides of two sub-polar triangles whose poles are isogonal conjugates determine the vertices of the diagonal triangle of the quadrangle formed by the intersections of the two conics inscribed in the triangle of reference which are the envelopes of the polars with respect to that triangle of points lying on the polars of the poles of the two sub-polar triangles.

* "Messenger of Mathematics," No. 451, November, 1908, p. 117.

The sides of the triangles PQR, P′Q′R′ pass through the vertices of the triangle ABC, the equations of QR and Q′R′ being respectively

$$\frac{\beta\beta_0}{qq'} + \frac{\gamma\gamma_0}{rr'} = o,$$

$$\frac{\beta}{\beta_0 qq'} + \frac{\gamma}{\gamma_0 rr'} = o.$$

The triangles PQR, P′Q′R′ are self-conjugate with respect to the following inscribed conic :—

$$\sqrt{\frac{\alpha}{p'}} + \sqrt{\frac{\beta}{q'}} + \sqrt{\frac{\gamma}{r'}} = o.$$

Any point on the conic

$$S_0 \equiv \alpha_0 \beta\gamma + \beta_0 \gamma\alpha + \gamma_0 \alpha\beta = o$$

is expressed by the co-ordinates

$$[-\kappa\alpha_0,\ \kappa\beta_0 + \gamma_0,\ \kappa(\kappa\beta_0 + \gamma_0)],$$

where κ is a variable parameter.

The equations of the sides EF, FD, DE of the triangle DEF are respectively

$$\kappa\beta_0\left(\frac{\alpha}{\alpha_0} + \frac{\beta}{\beta_0}\right) + \gamma_0\left(\frac{\gamma}{\gamma_0} + \frac{\alpha}{\alpha_0}\right) = o,$$

$$\kappa\beta_0\left(\frac{\alpha}{\alpha_0} + \frac{\beta}{\beta_0}\right) + \gamma_0\left(\frac{\gamma}{\gamma_0} - \frac{\alpha}{\alpha_0}\right) = o,$$

$$\kappa\beta_0\left(\frac{\alpha}{\alpha_0} - \frac{\beta}{\beta_0}\right) + \gamma_0\left(\frac{\gamma}{\gamma_0} + \frac{\alpha}{\alpha_0}\right) = o.$$

Hence EF passes through the point of intersection of the tangents to S_0 at B and C; FD passes through the point in which the line joining B to the pole of the conic meets the tangent at C; DE passes through the point in which the line joining C to the pole of the conic meets the tangent at B.

The sides of the triangles D′E′F′ are respectively

$$\kappa^2\beta_0\gamma + \kappa(\alpha_0\alpha + \beta_0\beta + \gamma_0\gamma) + \beta\gamma_0 = o,$$

$$\kappa^2\beta_0\gamma - \kappa(\alpha_0\alpha + \beta_0\beta - \gamma_0\gamma) - \beta\gamma_0 = o,$$

$$\kappa^2\beta_0\gamma + \kappa(\alpha_0\alpha - \beta_0\beta + \gamma_0\gamma) - \beta\gamma_0 = o,$$

and these lines envelop respectively the conics

$$(\alpha_0\alpha + \beta_0\beta + \gamma_0\gamma)^2 - 4\beta_0\gamma_0\beta\gamma = o,$$

$$(\alpha_0\alpha + \beta_0\beta - \gamma_0\gamma)^2 + 4\beta_0\gamma_0\beta\gamma = o,$$

$$(\alpha_0\alpha - \beta_0\beta + \gamma_0\gamma)^2 + 4\beta_0\gamma_0\beta\gamma = o$$

Hence the theorem,—

If the pole of a sub-polar triangle move on a conic circumscribing the triangle of reference, the sides of the sub-polar triangle will pass through three fixed points, while if the pole move on a straight line the sides of the sub-polar triangle will envelop three fixed conics.
